2022 Castro Valley Junior Math Tournament Solutions

1. 85 out of 150 Pteranodons brought shiny Dinomon cards, so $150 - 85 = 65$ Pteranodons didn't bring shiny Dinomon cards. Therefore, the probability is $\frac{65}{150} = \frac{13}{30}$.

2. To calculate the number of soccer balls that she can make, we divide the number of panels that she has by the number of panels needed for each ball. $\frac{490}{12} \approx 40.8$ and $\frac{921}{20} \approx 46.1$. Since she needs both types of panels to make the ball, we use the lesser of the two values, and we round down since it doesn't make sense to say that she made a fractional number of balls. The number of soccer balls that she can make is therefore 40.

3. Rate is amount divided by time, so the dinosaur can eat $\frac{6}{4.2}$ trees per day. Time is amount divided by rate, so it will take $\frac{10}{6/4.2} = \frac{10 \cdot 4.2}{6} = 7$ days for the dinosaur to eat 10 trees.

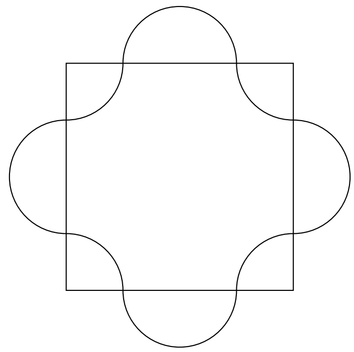
4. The sum of the interior angles of any simple quadrilateral is always $360°$. We are given the measure of two of the angles, and the measure of the angle opposite the $135°$ angle is $360° - 135° = 225°$. $30 + 95 + 225 + x = 360$, so $x = 360 - 30 - 95 - 225 = 10$.

5. Increasing one variable by 42% is the same as multiplying it by 1.42. Since the two variables are inversely proportional, when one is multiplied by 1.42, the other should be divided by 1.42. Dividing by 1.42 is the same as multiplying by $\frac{1}{1.42} \approx 0.704$. It follows that the percentage change of the other variable is $70.4\% - 100\% = -29.6\%$.

6. We can assign a number (formally called a random variable) to each college that is 1 if Cedric gets into that college and 0 otherwise. The number of colleges that Cedric gets into is the sum of the numbers for all 5 colleges. The expected value of the sum of multiple numbers is equal to the sum of the expected values of each number. If the probability of Cedric getting into a college is p, the expected value for that college is $1p + 0(1-p) = p$, so the expected value of the number of colleges that Cedric gets into is the sum of the probabilities of him getting into each college. The answer is therefore $0.8 + 0.05 + 0.2 + 0.9 + 0.7 \approx 2.7$.

7. In this problem, we meant to say that the second rock has the same shape as the first rock, but scaled up so that it is three times bigger in every direction. However, because the problem was ambiguous, we will also accept the answer that would be correct if the second rock is three times taller while having the same width. If the second rock is three times bigger in every direction, it would have $3^3 = 27$ times the volume of the first rock. Density is proportional to mass and inversely proportional to volume, so the density of the second rock is $10 \cdot \frac{3}{27} \approx 1.11$ grams per cubic centimeter. If the second rock had the same width as the first rock and is only three times taller, it would have three times the volume of the first rock. The density would be the same as the first rock since the mass and volume increased by the same factor.

8. We can divide the shape into a 4×4 square with four quarter-circles cut out of the corners and four semicircles added onto the sides. Since the area of a circle of radius 1 is π, the area of the middle region is $4^2 - 4 \cdot \frac{\pi}{4} = 16 - \pi$ and the combined area of the outer regions is $4 \cdot \frac{\pi}{2} = 2\pi$. Therefore, the total area is $16 - \pi + 2\pi = 16 + \pi \approx 19.14$.



9. This shape can be formed by slicing a cylinder of radius 3 and height $10 + 15 = 25$ in half along the slanted top face. The volume of a cylinder with radius 3 and height 25 is $\pi \cdot 3^2 \cdot 25 = 225\pi$, so the volume of the cylinder with the slanted top face is $\frac{225\pi}{2} \approx 353.4$.

10. Since Stella has a 0.5 probability of correctly predicting the coin flip, she and Henry have an equal probability of getting a point in each round. It follows that they have an equal probability of getting to 256 points first, so the answer is 0.5.

11. The lava pool initially has a radius of $\frac{8}{2} = 4$ meters and an area of $\pi \cdot 4^2 = 16\pi$ square meters. When the lava pool reaches Barry's position, it will have a radius of $4+15 = 19$ meters and an area of $\pi \cdot 19^2 = 361\pi$ square meters. The area increased by $361\pi - 16\pi = 345\pi$, so will take $\frac{345\pi}{0.1} = 3450\pi \approx 10838$ seconds.

12. First, we count the number of ways to form three pairs out of six objects in a line such that the two objects in each pair are separated by at most one other object. If we form a pair with the first two objects, then we could pair the third object with either the fourth or the fifth object. If we pair the first object with the third object, then the second object must be paired with the fourth object and the last two objects must form a pair. Therefore, there are 3 ways to form the pairs. For each pairing of objects, there are $3! = 6$ ways to assign the pairs of objects to pairs to pairs of shoes, and there are $2^3 = 8$ ways to choose the order of the two shoes within every pair. We multiply everything together to get $3 \cdot 6 \cdot 8 = 144$.

13. We can put the rows and columns into any order by swapping them around. Since all of the cells have different numbers, every permutation of rows and columns results in a different grid. There are $3! = 6$ possible permutations of the rows and $3! = 6$ possible permutations of the columns, so a total of $6 \cdot 6 = 36$ different grids can be obtained with these operations.

14. An elongated square gyrobicupola has 8 triangular faces and 18 square faces, for a total of $8 + 18 = 26$ faces. The probability of the first face being a triangle is $\frac{8}{26}$. If the first face is a triangle, the probability of the second face being adjacent to the first face is $\frac{3}{25}$, since 3 out of the 25 remaining faces are adjacent to the triangle. Similarly, the probability of the first face being a square is $\frac{18}{26}$, and if the first face is a square, the probability of the second face being adjacent to it is $\frac{4}{25}$. The overall probability of the second face being adjacent to the first face is then $\frac{8}{26} \cdot \frac{3}{25} + \frac{18}{26} \cdot \frac{4}{25} = \frac{48}{325}$.

15. If the average score on the four tests is at least 90, then the total score must be at least $90 \cdot 4 = 360$. The total score on the first three tests is $97 + 93 + 88 = 278$, so Alec needs to get at least $360 - 278 = 82$ on the fourth test. There are $100 - 80 + 1 = 21$ integers from 80 to 100 and 19 of them are at least 82, so the probability is $\frac{19}{21}$.

16. The triangle formed by the meteor, the begonia bush, and the blueberry bush is similar to the triangle formed by Barry's head, Barry's feet, and the blueberry bush by AA similarity since they share the angle at the blueberry bush and they both have a right angle. The side length ratios between similar triangles are the same, so $\frac{16}{40} = \frac{x}{x+37}$. Multiplying both sides by $x + 37$ and 40 gives $16(x + 37) = 40x$, so $16x + 16 \cdot 37 = 40x$, $592 = 24x$, and $x = \frac{592}{24} \approx 24.7$. Therefore, Barry was 24.7 meters away from the blueberry bush.

17. In order for there to be twice as many green dinosaurs as brown dinosaurs, the committee must have at least 8 green dinosaurs. If there are 8 green dinosaurs, then there are $\binom{10}{8} = 45$ ways to choose the green dinosaurs and $\binom{8}{4} = 70$ ways to choose the brown dinosaurs. If there are 9 green dinosaurs, then there are $\binom{10}{9} = 10$ ways to choose the green dinosaurs and $\binom{8}{3} = 56$ ways to choose the brown dinosaurs. If there are 10 green dinosaurs, then there are $\binom{10}{10} = 1$ way to choose the green dinosaurs and $\binom{8}{2} = 28$ ways to choose the brown dinosaurs. The number of ways to choose the committee is $45 \cdot 70 + 10 \cdot 56 + 1 \cdot 28 = 3738$.

18. In order for Andrew to end at least six meters to the right of his starting point, he must get heads at least four times. There are five ways to get exactly four heads, and one way to get five heads. The probability of every possible sequence of outcomes is $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$, so the probability of getting at least four heads is $\frac{5+1}{32} = \frac{3}{16}$. This is the probability that Andrew ends at least six meters to the right of his starting point.

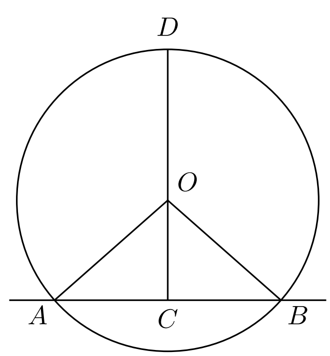
19. By definition, every possible value of $\gcd(k, 360)$ is a factor of 360, and for each factor a of 360, $\gcd(a, 360) = a$. Therefore, the possible values of x are the factors of 360. The answer is the smallest factor of 360 that is at least 75, which is 90.

20. If Julia rolls a 1 or 2 the first time, the expected value of the second die roll is $\frac{1+2+3+4+5+6}{6} = \frac{7}{2}$. If she rolls 3, 4, 5, or 6 the first time, the expected value of the second die roll is $\frac{2+2+4+4+4+4}{6} = \frac{10}{3}$. To get the expected value of the product, we multiply each possible outcome of the first die roll with the corresponding expected value for the second die roll and compute the mean to get

$$\frac{1 \cdot \frac{7}{2} + 2 \cdot \frac{7}{2} + 3 \cdot \frac{10}{3} + 4 \cdot \frac{10}{3} + 5 \cdot \frac{10}{3} + 6 \cdot \frac{10}{3}}{6} = 11.75$$

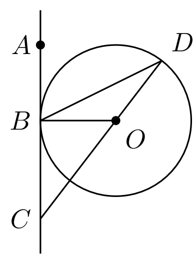
21. During the fourteen-day period, the population tripled twice, so it is multiplied by $3^2 = 9$. This means that it increased by 8 times the size of the population at the start of the period. This is equal to 72, so the number of dinosaurs at the start of the period is $\frac{72}{8} = 9$.

22. I will write all lengths in centimeters. In the following figure, $\overline{AB}$ represents the ground level and the circle represents the ball. $\overline{CD}$ is a vertical line going through the center of the hole and the ball. $CB = 75$, since that is the radius of the hole. $OB = 100$, since that is the radius of the ball. Using the Pythagorean theorem, we get $OC = \sqrt{100^2 - 75^2} = \sqrt{4375}$. To get the vertical distance from the ground to the top of the sphere, we add the length of $\overline{OD}$, which is the radius of the ball. Therefore, the answer is $\sqrt{4375} + 100 \approx 166.1$.

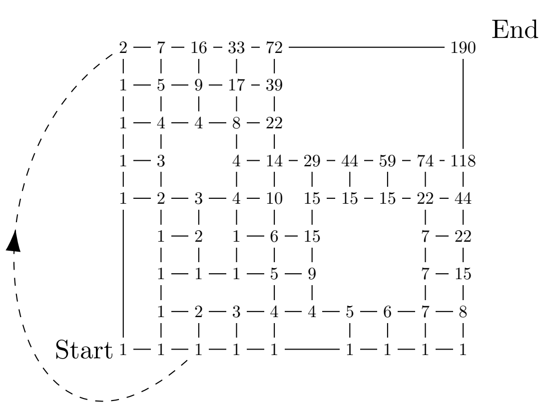


23. If we treat the two scores as coordinates for a point on the plane, each combination of scores can be represented by a point in the square formed by the lines $x = 60$, $x = 100$, $y = 60$, and $y = 100$. In order for the average scores to be at least 90, the sum of the two scores have to be at least $2 \cdot 90 = 180$, so the point has to be above the line $x + y = 180$. If you graphed the square and the line, you will see that the region inside the square and above the line is a triangle with one eighth the area of the square. Therefore, the probability that the average scores is at least 90 is $\frac{1}{8}$.

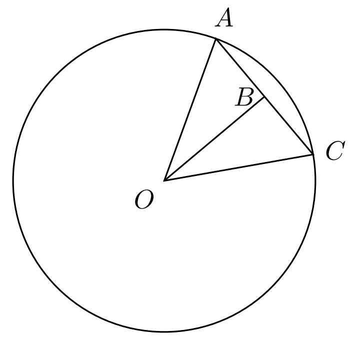
24. First, we draw a line from point B to point O. This line is perpendicular to $\overline{AC}$ since $\overline{AC}$ is the tangent line of the circle at point B. $\angle BCO$ is the same as $\angle ACD$ and $\angle CBO$ is a right angle, so we can use the triangle angle sum theorem to find the measure of $\angle BOC$, which is $180° - 28° - 90° = 62°$. We can find the measure of $\angle BOD$ by subtracting from 180° to get $180° - 62° = 118°$. $\triangle BOD$ is isosceles, since both $\overline{BO}$ and $\overline{DO}$ are radii of the circle and therefore have the same length. The sum of the measures of $\angle OBD$ and $\angle ODB$ is $180° - 118° = 62°$ by the triangle angle sum theorem, so the measure of $\angle OBD$ is $\frac{62°}{2} = 31°$. Lastly, $\angle ABO$ is a right angle, so we subtract the measure of $\angle OBD$ from 90° to get $90° - 31° = 59°$.



25. We can define the value of an intersection to be the number of paths from the starting point to that intersection. The value of the starting point is 1, since the only way to reach the starting point is to do nothing. We can calculate the value of any other intersection by considering the possible last steps of all the paths to that intersection. If there is only one street going into an intersection, then the value of that intersection is the same as the value of the previous intersection on that street. If there are two streets going into an intersection, then the value of that intersection is the sum of the values of the two possible previous intersections. We start from the bottom left and compute the values of every intersection using the values of other intersections that we already computed until we reach the end intersection. When we compute the value of the top left intersection, we take the sum of the value of the intersection below it and the value of the intersection at the starting point of the subway. The value of the end intersection is the answer.

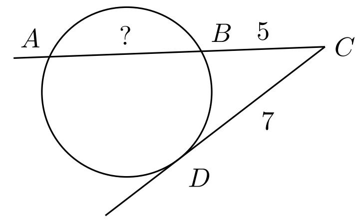


26. Let's say we picked one of the points first. We will calculate the position of the second point such that the distance from the center of the circle to the chord between the two points is exactly $\sqrt{3}$. In the following figure, points A and C are the two points on the circle, and $\overline{OB}$ is the line from the center of the circle to the closest point on the chord $\overline{AC}$. $OB = \sqrt{3}$ and $\angle OBC$ is a right angle since B is the point on the chord that is closest to point O. $OC = 2$, since it is a radius of the circle. The ratio between OB and OC shows that $\triangle OBC$ is a 30-60-90 triangle, so $\angle BOC$ has a measure of 30°. $\angle BOA$ also has a measure of 30°, so the measure of $\angle AOC$, which is also the measure of the arc between the two points on the circle, is 60°. In order for the distance from the center of the circle to the chord to be at least $\sqrt{3}$, the measure the arc between the two points has to be at most 60°. If the position of the first point is fixed, then the possible positions of the second point forms a 120° arc centered on the first point. This is a third of the circle, so the probability that the chord is at least $\sqrt{3}$ away from the center of the circle is $\frac{1}{3}$.



27. Let the original number be x. Removing the first digit is the same as subtracting 7000000, and adding it back at the end is the same as multiplying by 10 and adding 7. Therefore, we have $10(x - 7000000) + 7 - x = 1691973$. Expanding and simplifying the left side gives $10x - 70000000 + 7 - x = 9x - 69999993 = 1691973$, so $x = \frac{1691973+69999993}{9} = 7965774$.

28. The tangent-secant theorem, which is a special case of the power of a point, tells us that $AC \cdot BC = DC^2$. Therefore, $AC \cdot 5 = 7^2$ and $AC = 9.8$. To find the missing length, we subtract the length of $\overline{BC}$ to get $9.8 - 5 = 4.8$.



29. The sum of two numbers with a constant product is minimized when the numbers are close together. $10^{1000} - 1$ can be factored into $10^{500} - 1$ and $10^{500} + 1$ using the difference of squares, and there is no way to factor $10^{1000} - 1$ into two numbers that are closer together since $10^{500} \cdot 10^{500} \neq 10^{1000} - 1$. The sum of $10^{500} - 1$ and $10^{500} + 1$ is $2 \cdot 10^{500}$, which has 501 digits since it is 2 followed by 500 zeroes.

30. The Euclidean algorithm tells us that when computing the GCD, subtracting one of the numbers from the other doesn't change the answer. Therefore, $\gcd(2a + 3b, a + 2b) = \gcd(a + b, a + 2b) = \gcd(a + b, b) = \gcd(a, b) = 12$. The LCM of two numbers is equal to their product divided by their GCD, so $\operatorname{lcm}(ab^2, a^2b) = \frac{a^3b^3}{\gcd(ab^2,a^2b)} = \frac{a^3b^3}{ab\gcd(b,a)} = \frac{a^2b^2}{\gcd(a,b)}$. We know the GCD is 12, so $\frac{a^2b^2}{12} = 1881792$, and $ab = \sqrt{12 \cdot 1881792} = 4752$.